Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

For any natural m and any real $a,b,c>0$ find $\lim_{n\to\infty}\prod_{k=n}^{nm}\frac{ak+b}{ak+c}$.

Solution.

Note that $\prod_{k=n}^{nm}\frac{ak+b}{ak+c}=\prod_{k=n}^{nm}\frac{1+\frac{b}{ak}}{1+\frac{c}{ak}}=\frac{\prod_{k=n}^{nm}\left(1+\frac{b}{ak}\right)}{\prod_{k=n}^{nm}\left(1+\frac{c}{ak}\right)}$.

For any positive real r we will find $\lim_{n\to\infty}\prod_{k=n}^{nm}\left(1+\frac{r}{k}\right)$.

Since $e^{\frac{r}{n+r}}<1+\frac{r}{n}<e^{\frac{r}{n}}$ (follows from $\left(1+\frac{r}{n}\right)^n<e^r<\left(1+\frac{r}{n}\right)^{n+r}$)* then

$$e^{\sum_{k=n}^{nm}\frac{r}{k+r}}<\prod_{i=n}^{nm}\left(1+\frac{r}{k}\right)<e^{\sum_{k=n}^{nm}\frac{r}{k}}.$$

Let $p:=\lceil r\rceil$. Then $\sum_{k=n}^{nm}\frac{1}{k}=h_{nm}-h_{n-1}$, $\sum_{k=n}^{nm}\frac{1}{k+r}\geq\sum_{k=n}^{nm}\frac{1}{k+p}=h_{nm+p}-h_{n+p-1}$, where $h_n:=1+\frac{1}{2}+\ldots+\frac{1}{n},n\in\mathbb{N}$ and, therefore,

$$e^{r(h_{nm+p}-h_{n+p-1})}<\prod_{i=n}^{nm}\left(1+\frac{r}{k}\right)<e^{r(h_{nm}-h_{n-1})}.$$

Since** $\lim_{n\to\infty}(h_{nm+k}-h_{n+k-1})=\lim_{n\to\infty}(h_{nm}-h_{n-1})=\ln m$ then $\lim_{n\to\infty}\prod_{i=n}^{nm}\left(1+\frac{r}{i}\right)=e^{r\ln m}=m^r$.

Hence, $\lim_{n\to\infty}\prod_{k=n}^{nm}\frac{ak+b}{ak+c}=\frac{\lim_{n\to\infty}\prod_{k=n}^{nm}\left(1+\frac{b}{ak}\right)}{\lim_{n\to\infty}\prod_{k=n}^{nm}\left(1+\frac{c}{ak}\right)}=\frac{m^{\frac{b}{a}}}{m^{\frac{c}{a}}}=m^{\frac{b-c}{a}}$.

Complements.

*. Substitution $u:=-\frac{1}{1+t},t>0$ in inequality $e^u\geq 1+u,\ u\in\mathbb{R}$ gives us

$e^{-\frac{1}{1+t}}>1-\frac{1}{1+t}\iff e^{-\frac{1}{1+t}}>\frac{t}{1+t}\iff e^{\frac{1}{1+t}}<1+\frac{1}{t}$ and for $t=\frac{x}{r},x>0$

we obtain $e^{\frac{r}{x+r}}<1+\frac{r}{t}$.

**. Since $\ln n+\gamma<h_n<\ln(n+1)+\gamma$, where γ –Euler's constant then we obtain

$\ln m<h_{nm}-h_{n-1}<\ln\frac{nm+1}{n-1}$,

$\frac{nm+k}{n+k}<h_{nm+k}-h_{n+k-1}<\ \ln\frac{nm+k+1}{n+k-1}$ and

$$\lim_{n\to\infty}\ln\frac{nm+k}{n+k}=\lim_{n\to\infty}\ln\frac{nm+k+1}{n+k-1}=\lim_{n\to\infty}\ln\frac{nm+1}{n-1}=m.$$